

# SEMISTABILITY CRITERION FOR PARABOLIC VECTOR BUNDLES ON CURVES

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**ABSTRACT.** We give a cohomological criterion for a parabolic vector bundle on a curve to be semistable. It says that a parabolic vector bundle  $\mathcal{E}_*$  with rational parabolic weights is semistable if and only if there is another parabolic vector bundle  $\mathcal{F}_*$  with rational parabolic weights such that the cohomologies of the vector bundle underlying the parabolic tensor product  $\mathcal{E}_* \otimes \mathcal{F}_*$  vanish. This criterion generalizes the known semistability criterion of Faltings for vector bundles on curves and significantly improves the result in [Bis07].

## 1. INTRODUCTION

We will work over an algebraically closed ground field of characteristic zero.

Let  $X$  be an irreducible smooth projective curve. A theorem due to Faltings says that a vector bundle  $E$  over  $X$  is semistable if and only if there is a vector bundle  $F$  over  $X$  such that  $H^0(X, E \otimes F) = 0 = H^1(X, E \otimes F)$  (see [Fal93, p. 514, Theorem 1.2] and [Fal93, p. 516, Remark]). Let  $D$  be a reduced effective divisor on  $X$ . For a parabolic vector bundle  $W_*$  on  $X$  with parabolic divisor  $D$ , the underlying vector bundle will be denoted by  $W_0$ ; see [MS80], [MY92] for parabolic vector bundles. Let  $r$  be a positive integer. Denote by  $\text{Vect}(X, D, r)$  the category of parabolic vector bundles on  $X$  with parabolic structure along  $D$  and parabolic weights being integral multiples of  $1/r$ . In [Bis07] the following theorem was proved:

**Theorem 1.1.** *There is a parabolic vector bundle  $\mathcal{V}_* \in \text{Vect}(X, D, r)$  with the following property: A parabolic vector bundle  $\mathcal{E}_*$  is semistable if and only if there is a parabolic vector bundle  $\mathcal{F}_* \in \text{Vect}(X, D, r)$  with  $H^0(X, (\mathcal{E}_* \otimes \mathcal{V}_* \otimes \mathcal{F}_*)_0) = 0 = H^1(X, (\mathcal{E}_* \otimes \mathcal{V}_* \otimes \mathcal{F}_*)_0)$ , where  $(\mathcal{E}_* \otimes \mathcal{V}_* \otimes \mathcal{F}_*)_*$  is the parabolic tensor product.*

Theorem 1.1 was also proved in [Par10]. It should be mentioned that the vector bundle  $\mathcal{V}_*$  in Theorem 1.1 is not canonical; it depends upon the choice of a suitable ramified Galois covering  $Y \rightarrow X$  that transforms parabolic bundles in  $\text{Vect}(X, D, r)$  into  $G$ -linearized vector bundles on  $Y$ , where  $G$  is the Galois group for the covering. However, many different covers do this.

We prove that  $\mathcal{V}_*$  in Theorem 1.1 can be chosen to be the trivial line bundle  $\mathcal{O}_X$  equipped with the trivial parabolic structure. More precisely, we prove the following theorem (see Theorem 6.1):

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**Theorem 1.2.** *A parabolic vector bundle  $\mathcal{E}_* \in \text{Vect}(X, D, r)$  is semistable if and only if there is a parabolic vector bundle  $\mathcal{F}_* \in \text{Vect}(X, D, r)$  such that*

$$H^0(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0) = 0 = H^1(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0).$$

Theorem 1.2 is proved by systematically working with stacks. Compare this method with the earlier attempts (cf. [Bis07], [Par10]) that landed in the weaker version given in Theorem 1.1.

## 2. PARABOLIC BUNDLES AND ROOT STACKS

Recall that to give a morphism  $X \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$  is the same as giving a line bundle  $\mathcal{L}$  with section  $s$  on  $X$  (see [Cad07]). Given a positive integer  $r$ , there is a natural morphism

$$\theta_r : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$$

defined by  $t \mapsto t^r$ , with  $t \in \mathbb{A}^1$ . We define the root stack  $X_{(\mathcal{L}, s, r)}$  to be the fibered product

$$X \times_{[\mathbb{A}^1/\mathbb{G}_m], \theta_r} [\mathbb{A}^1/\mathbb{G}_m].$$

When the section is non-zero, this root stack is an orbifold curve; see [Cad07, Example 2.4.6].

The data  $(\mathcal{L}, s)$  corresponds to an effective divisor  $D$  on  $X$ . We will henceforth assume that this divisor is reduced. Sometime we write  $X_{D, r}$  instead of  $X_{\mathcal{L}, s, r}$ .

We think of the ordered set  $\frac{1}{r}\mathbb{Z}$  of rational numbers with denominator  $r$  as a category. Let  $j$  be an integer multiple of  $1/r$ . Given a functor from the opposite category

$$\mathcal{F}_* : \left(\frac{1}{r}\mathbb{Z}\right)^{\text{op}} \rightarrow \text{Vect}(X),$$

we denote by  $\mathcal{F}_*[j]$  its shift by  $j$ , so

$$\mathcal{F}_i[j] = \mathcal{F}_{i+j}.$$

There is a natural transformation  $\mathcal{F}_*[j] \rightarrow \mathcal{F}_*$  when  $j \geq 0$ .

A *vector bundle with parabolic structure* over  $D$  such that the parabolic weights are integral multiples of  $1/r$  is a functor

$$\mathcal{F}_* : \left(\frac{1}{r}\mathbb{Z}\right)^{\text{op}} \rightarrow \text{Vect}(X)$$

together with a natural isomorphism

$$j : \mathcal{F}_* \otimes \mathcal{O}_X(-D) \xrightarrow{\sim} \mathcal{F}[1]$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}_* \otimes \mathcal{O}_X(-D) & \longrightarrow & \mathcal{F}[1] \\ & \searrow & \downarrow \\ & & \mathcal{F}_* \end{array}$$

(see [MY92], [MS80]). The *underlying vector bundle* of a parabolic vector bundle is the value of this functor at 0. We have previously denoted this by  $\mathcal{F}_0$ . For a functor  $\mathcal{F}_*$  defining a parabolic vector bundle, the value of  $\mathcal{F}_*$  at  $t \in \frac{1}{r}\mathbb{Z}$  will be denoted by  $\mathcal{F}_t$ .

Denote by  $\text{Vect}(X, D, r)$  the category of vector bundles on  $X$  with parabolic structure along  $D$  and parabolic weights integral multiples of  $1/r$ . It is a tensor category.

**Theorem 2.1.** *There is an equivalence of tensor categories*

$$F : \text{Vect}(X_{(\mathcal{L}, s, r)}) \xrightarrow{\sim} \text{Vect}(X, D, r).$$

*The equivalence preserves parabolic degree and semistability (see § 4 below).*

The functor  $F$  has the following explicit description. There is a natural root line bundle  $\mathcal{N}$  on  $X_{(\mathcal{L}, s, r)}$ . Given a vector bundle  $\mathcal{F}$  on the root stack, the corresponding parabolic bundle is the functor defined by

$$l/r \mapsto \pi_*(\mathcal{F} \otimes \mathcal{N}^l).$$

*Proof of Theorem 2.1.* See [Bor07, Section 3] and [Bis97].  $\square$

### 3. ROOT STACKS AS QUOTIENT STACKS

For the map  $z \mapsto z^n$  defined around  $0 \in \mathbb{C}$ , the *ramification index* at 0 will be  $n - 1$ .

We will need the following theorem :

**Theorem 3.1.** *Suppose  $k = \mathbb{C}$ . There is a finite Galois covering  $Y \rightarrow X$  ramified over  $D$  with ramification index  $r - 1$  at each point in  $D$  if and only if either  $X \neq \mathbb{P}^1$  or  $X = \mathbb{P}^1$  with  $|D| \neq 1$ .*

*Proof.* See [Nam87, p. 29, Theorem 1.2.15].  $\square$

**Corollary 3.2.** *Theorem 3.1 holds over any algebraically closed ground field of characteristic zero.*

*Proof.* This follows from [SGA1, Expose IX, Theorem 4.10]. See also Proposition 7.2.2 in [Mur67, p. 146].  $\square$

**Proposition 3.3.** *Suppose that either  $X \neq \mathbb{P}^1$  or  $|D| \neq 1$ . Then  $X_{(D, r)}$  is a quotient stack.*

*Proof.* Fix a covering  $Y \rightarrow X$  as in Corollary 3.2. Let  $G$  be the Galois group for this covering. Our goal is to show that  $X_{(D, r)} = [Y/G]$ .

Let  $R$  be the ramification divisor in  $Y$ . Then the reduced divisor  $R_{\text{red}}$  produces a morphism

$$(1) \quad Y \rightarrow X_{(D, r)}$$

via the universal property of root stacks. As  $R_{\text{red}}$  is  $G$ -invariant so is the morphism in (1). Hence we obtain a morphism

$$[Y/G] \rightarrow X_{(D, r)}.$$

To show that this morphism is an isomorphism is a local condition for the flat topology and follows from [Cad07, Example 2.4.6].  $\square$

## 4. SEMISTABILITY

Recall that the *parabolic degree* of a parabolic vector bundle  $\mathcal{E}_*$  over  $X$  is defined to be

$$\begin{aligned} \deg_{\text{par}}(\mathcal{E}_*) &:= \text{rk}(\mathcal{E}_0)(\deg D - \chi(\mathcal{O}_X)) + \frac{1}{r}(\sum_{i=1}^r \chi(\mathcal{E}_{i/r})) \\ &= \text{rk}(\mathcal{E}_0) \deg D + \frac{1}{r} \sum_{i=1}^r \deg(\mathcal{E}_{i/r}) \end{aligned}$$

(see [MS80], [Bis97], [Bor07, § 4]). The *slope* is defined as usual :

$$\mu(\mathcal{E}_*) := \frac{\deg_{\text{par}}(\mathcal{E})}{\text{rk}(\mathcal{E})}.$$

A parabolic vector bundle  $\mathcal{E}_*$  is said to be *semistable* if

$$\mu(\mathcal{E}_*) \geq \mu(\mathcal{F}_*)$$

for all parabolic subbundles  $\mathcal{F}_*$ .

*Example 4.1.* Let us describe all the parabolic semistable bundles on  $\mathbb{P}^1$  with one parabolic point, meaning  $D = x$ , where  $x$  is some point on  $\mathbb{P}^1$ . Let  $\mathcal{E}_*$  be a semistable parabolic vector bundle. Then we may write

$$\mathcal{E}_0 = \bigoplus_{k=1}^m \mathcal{O}(n_k)^{s_k}$$

[Gro57]. We may assume that the integers  $n_i$  are strictly decreasing. A subbundle  $\mathcal{F}_*$  is defined by taking

$$\mathcal{F}_{i/r} = \mathcal{O}(n_1)^{s_1} \cap \mathcal{E}_{i/r}$$

for  $0 \leq i < r$ . This extends to a parabolic subbundle of  $\mathcal{E}_*$ . We see immediately that

$$\mu(\mathcal{F}_*) > \mu(\mathcal{E}_*)$$

when  $m > 1$ . Consequently, a parabolic vector bundle  $\mathcal{E}_*$  of rank  $n$  over  $\mathbb{P}^1$  with one parabolic point is semistable if and only if

$$\mathcal{E}_* = (\mathcal{L}_*)^{\oplus n},$$

where  $\mathcal{L}_*$  is a parabolic line bundle.

## 5. GROTHENDIECK-RIEMANN-ROCH THEOREM FOR DELIGNE-MUMFORD STACKS

In this section we recall the pertinent results from [Tö99]. An excellent summary of this paper of Töen can be found in the appendix to [Bor07]. We denote by  $\mathfrak{X}$  a smooth Deligne-Mumford stack that is proper over our ground field  $k$ . We equip it with the étale topology. The category of vector bundles (respectively, coherent sheaves) on  $\mathfrak{X}$  is an exact category so we may form the groups

$$K_i(\mathfrak{X}) \quad (\text{respectively, } G_i(\mathfrak{X})).$$

Let  $\mathcal{K}_i$  denote the sheaf in the étale topology on  $\mathfrak{X}$  associated to the presheaf

$$(X \longrightarrow \mathfrak{X}) \longmapsto K_i(X).$$

Set

$$H^i(\mathfrak{X}, \mathbb{Q}) = H^i(\mathfrak{X}, \mathcal{K}_i \otimes \mathbb{Q}).$$

By [Gil81] we have Chern classes and hence Chern characters and Todd classes

$$c_i^{\text{et}}, \text{ch}^{\text{et}}, \text{td}^{\text{et}} : K_0(\mathfrak{X}) \longrightarrow H^*(\mathfrak{X}).$$

Let  $I_{\mathfrak{X}} := \mathfrak{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$  be the inertia stack of  $\mathfrak{X}$ . Let  $\mu_{\infty}$  denote the group of roots of unity in  $\overline{\mathbb{Q}}$ , and set  $\Lambda := \mathbb{Q}(\mu_{\infty})$ . If  $\mathcal{G}$  is a locally free sheaf on  $I_{\mathfrak{X}}$ , the inertial action induces an eigenspace decomposition

$$\mathcal{G} = \bigoplus_{\zeta \in \mu_{\infty}} \mathcal{G}^{(\zeta)}.$$

Let

$$\rho_{\mathfrak{X}} : K_0(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda \longrightarrow K_0(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda$$

be the morphism defined by

$$\mathcal{G} \longmapsto \sum \zeta [\mathcal{G}^{(\zeta)}].$$

We have a morphism, called the *Frobenius character*,

$$\phi_{\mathfrak{X}} : K_0(\mathfrak{X}) \otimes_{\mathbb{Z}} \Lambda \xrightarrow{\pi_{\mathfrak{X}}^*} K_0(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda \xrightarrow{\rho_{\mathfrak{X}}} K_0(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda \longrightarrow K_{0,\text{et}}(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda.$$

The ring  $K_0$  is a lambda ring and we write  $\lambda_{-1}(x) = \sum (-1)^i \lambda_i(x)$ . Define

$$\alpha_{\mathfrak{X}} := \rho_{\mathfrak{X}}(\lambda_{-1}([\Omega_{I_{\mathfrak{X}}/\mathfrak{X}}^1])) \in K_{0,\text{et}}(I_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \Lambda.$$

Finally define the characteristic classes

$$\text{ch}^{\text{rep}}(x) := \text{ch}^{\text{et}}(\phi_{\mathfrak{X}}(x))$$

and

$$\text{td}^{\text{rep}}(\mathfrak{X}) := \text{ch}^{\text{et}}(\alpha_{\mathfrak{X}}^{-1}) \text{td}^{\text{et}}(\mathcal{T}_{I_{\mathfrak{X}}}).$$

**Theorem 5.1.** *Denote by  $\int_{\mathfrak{X}}^{\text{rep}}$  the push-forward  $p_*$  for  $p : I_{\mathfrak{X}} \longrightarrow \text{Spec}(k)$ . The following holds:*

$$\chi(\mathfrak{X}, \mathcal{F}) = \int_{\mathfrak{X}}^{\text{rep}} \text{td}^{\text{rep}}(\mathfrak{X}) \text{ch}^{\text{rep}}(\mathcal{F}).$$

*Proof.* See [Tö99, Corollary 4.13]. □

**Corollary 5.2.** *Suppose that  $\mathfrak{X}$  is a proper orbifold curve. Then*

$$\mu(\mathcal{F}) = \chi(\mathcal{F}) - \int_{\mathfrak{X}}^{\text{rep}} \text{td}^{\text{rep}}(\mathfrak{X}).$$

*Proof.* We have that  $\pi_{\mathfrak{X}}^*(\mathcal{F})$  is an eigensheaf with eigenvector 1 as the stack  $\mathfrak{X}$  is generically a variety. There is a diagram

$$\begin{array}{ccc} I_{\mathfrak{X}} & & \\ \downarrow \pi_{\mathfrak{X}} & \searrow p_I & \\ \mathfrak{X} & \xrightarrow{p} & \text{Spec}(k). \end{array}$$

By the projection formula,

$$p_{I,*}(c_1^{\text{et}}(\pi_{\mathfrak{X}}^*(\mathcal{F}))) = p_*(c_1^{\text{et}}(\mathcal{F})).$$

In view of Theorem 5.1, the result follows from the fact that  $\deg(\mathcal{F}) = p_*(c_1^{\text{et}}(\mathcal{F}))$  ([Bor07, Theorem 4.3]) and the usual expression for the Chern character. □

**Corollary 5.3.** *Suppose that there is a vector bundle  $\mathcal{E}$  so that  $H^i(\mathfrak{X}, \mathcal{E} \otimes \mathcal{F}) = 0$  for  $i = 0, 1$ . Then  $\mathcal{F}$  is semistable.*

*Proof.* Suppose there is a subsheaf  $\mathcal{F}'$  of  $\mathcal{F}$  with

$$\mu(\mathcal{F}') > \mu(\mathcal{F}).$$

Then it follows from Corollary 5.2 that

$$\frac{\chi(\mathcal{E} \otimes \mathcal{F}')}{\text{rank}(\mathcal{E} \otimes \mathcal{F}')} - \frac{\chi(\mathcal{E} \otimes \mathcal{F})}{\text{rank}(\mathcal{E} \otimes \mathcal{F})} > 0.$$

Since  $\chi(\mathcal{E} \otimes \mathcal{F}) = 0$ , this implies that  $H^0(\mathfrak{X}, \mathcal{E} \otimes \mathcal{F}') \neq 0$ . But  $\mathcal{E} \otimes \mathcal{F}' \subset \mathcal{E} \otimes \mathcal{F}$ . Hence  $H^0(\mathfrak{X}, \mathcal{E} \otimes \mathcal{F}) \neq 0$  which is a contradiction  $\square$

## 6. SEMISTABILITY CRITERION

**Theorem 6.1.** *A vector bundle with parabolic structure  $\mathcal{E}_* \in \text{Vect}(X, D, r)$  is semistable if and only if there is a parabolic vector bundle  $\mathcal{F}_* \in \text{Vect}(X, D, r)$  with*

$$H^i(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0) = 0$$

for all  $i$ , where  $(\mathcal{E}_* \otimes \mathcal{F}_*)_*$  is the parabolic tensor product.

*Proof.* We have a morphism  $\pi : X_{D,r} \rightarrow X$ , and  $\pi_*$  is exact as  $\text{char}(k) = 0$ . Hence by the Leray spectral sequence,

$$H^i(X, \pi_*(\mathcal{F})) = H^i(X_{D,r}, \mathcal{F})$$

for all  $i$ .

Suppose that there is a parabolic vector bundle  $\mathcal{F}_* \in \text{Vect}(X, D, r)$  with

$$H^0(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0) = 0 = H^1(X, (\mathcal{E}_* \otimes \mathcal{F}_*)_0).$$

Applying Theorem 2.1, we deduce from Corollary 5.3 that  $\mathcal{E}_*$  is semistable.

To prove the converse, assume that  $\mathcal{E}_*$  is semistable. We break up into two cases.

The case of  $\mathbb{P}^1$  with exactly one parabolic point: Applying Example 4.1, we see that

$$\mathcal{E}_0 = \bigoplus \mathcal{O}(n)^m.$$

So tensoring with  $\mathcal{O}(-n-1)$  does the job.

All other cases: In view of Proposition 3.3 we may assume that we have a quotient stack, so  $X_{D,r} = [Y/G]$ . Then given a semistable parabolic bundle on  $X$ , we obtain a corresponding semistable  $G$ -linearized vector bundle  $\mathcal{E}$  on  $Y$ . We note that this implies that the vector bundle  $\mathcal{E}$  is semistable [Bis97, p. 308, Lemma 2.7]. By [Fal93, p. 514, Theorem 1.2], there is a vector bundle  $\mathcal{F}$  on  $Y$  such that all the cohomology groups of  $\mathcal{F} \otimes \mathcal{E}$  vanish. Consider

$$\tilde{\mathcal{F}} = \bigoplus_{g \in G} g^* \mathcal{F}.$$

The vector bundle  $\tilde{\mathcal{F}}$  has a natural  $G$ -action and

$$H^i(Y, \tilde{\mathcal{F}} \otimes \mathcal{E}) = 0$$

for all  $i$ . The vector bundle  $\tilde{\mathcal{F}}$  produces a vector bundle on  $[Y/G]$ , which will also be denoted by  $\tilde{\mathcal{F}}$ . Finally

$$H^i([Y/G], \tilde{\mathcal{F}} \otimes \mathcal{E}) = H^i(Y, \tilde{\mathcal{F}} \otimes \mathcal{E})^G = 0.$$

The theorem now follows.  $\square$

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## REFERENCES

- [Bis97] I. Biswas, Parabolic bundles as orbifold bundles, *Duke Math. Jour.* **88** (1997), 305–32.
- [Bis07] I. Biswas, A cohomological criterion for semistable parabolic vector bundles on a curve, *C. R. Math. Acad. Sci. Paris* **345** (2007), 325–328.
- [Bor07] N. Borne, Fibrés paraboliques et champ des racines, *Int. Math. Res. Not.*, Art. ID rnm049, **38**, (2007).
- [Cad07] C. Cadman, Using stacks to impose tangency conditions on curves, *Amer. Jour. Math.* **129** (2007), 405–427.
- [Fal93] G. Faltings, Stable  $G$ -bundles and projective connections, *Jour. Algebraic Geom.* **2** (1993), 507–568.
- [Gil81] H. Gillet, Riemann-Roch theorems for higher algebraic  $K$ -theory, *Adv. in Math.* **40** (1981), 203–289.
- [Gro57] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, *Amer. Jour. Math.* **79** (1957), 121–138.
- [MY92] M. Maruyama and K. Yokogawa, Moduli of parabolic stable sheaves, *Math. Ann.* **293** (1992), 77–99.
- [MS80] V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* **248** (1980), 205–239.
- [Mur67] J. P. Murre, *Lectures on an introduction to Grothendieck’s theory of the fundamental group*, Tata Institute of Fundamental Research, Bombay, 1967, Tata Institute of Fundamental Research Lectures on Mathematics, No 40.
- [Nam87] M. Namba, *Branched coverings and algebraic functions*, Pitman Research Notes in Mathematics Series, Vol. 161, Longman Scientific & Technical, Harlow, 1987.
- [Par10] A. J. Parameswaran, Parabolic coverings I: the case of curves, *Jour. Ramanujan Math. Soc.* **25** (2010), 233–251.
- [SGA1] Revêtements étales et groupe fondamental (SGA 1), Séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck, with two papers by M. Raynaud. Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224] Société Mathématique de France 2003.
- [Tö99] B. Töen, Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford, *K-Theory* **18** (1999), 33–76.

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